THE 2-MODULAR PERMUTATION MODULES ON FIXED POINT FREE INVOLUTIONS OF SYMMETRIC GROUPS

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1.1 Let $A = kG$, the group algebra of some finite group where the characteristic of the field k divides $|G|$. In contrast to working over the complex field, the kG-modules are not usually semisimple. If a Sylow p-subgroup of G is not cyclic then there are infinitely many indecomposable kG-modules, and we usually enjoy little control over the category of such modules. It is therefore instructive to find classes of modules which may be expressed as a sum of a not very great number of indecomposables, and to understand the structure of these indecomposables. Permutation kG-modules, and their indecomposable summands (called p-permutation modules), provide one such class.

In [2] we studied permutation modules for symmetric groups acting on conjugacy classes of fixed point free elements which are products of q-cycles, and where k has characteristic p. With each component (that is, indecomposable summand) we associated a fixed point set, and we obtained a general description for the fixed point sets. In this paper we shall complete the analysis of the case $q = p = 2$. Thus we shall determine the fixed point sets of the components of the permutation module of the symmetric group $\mathrm{Sym}(2n)$ on the set of its fixed point free involutions. At the same time we find the vertex and Brauer quotient for each component, and the ordinary character associated with each component.

Each component M of a permutation kG-module determines two invariants, namely a vertex Q (up to conjugation) and a projective indecomposable p-permutation module for for $N_G(Q)/Q$; these invariants characterise M up to isomorphism. These invariants were introduced by Green [4] and adapted to permutation modules by Puig; see the paper of Broué [1].

Given an indecomposable kG-module M, a vertex of M is a subgroup $H \leq G$ minimal subject to satisfying the condition that there be a kH-module L for which M is a component of $\mathrm{Ind}_H^G(L)$. This vertex is unique up to conjugation in G. Any Sylow p-subgroup of G satisfies this condition and so vertices are always p-subgroups. The kH-module L is called a source of M and is unique up to conjugation in $N_G(H)$. However, M is a p-permutation module if and only if M is a component of a permutation module and this happens exactly when L is the trivial kH-module. Hence p-permutation modules are also known as trivial source modules.

The Brauer quotient $M(H)$ of M is a module for the group $N_G(H)/H$. When M is a component of a permutation module $k\Omega$ then $M(H)$ is a component of $k\mathrm{Fix}_\Omega H$,

hence is a p-permutation module. Broué's correspondence (1.1) asserts a 1-1 correspondence between the projective components of $k\mathrm{Fix}_\Omega H$ and the components of M with vertex H.

In [2] we classified the possible fixed point sets when the group is a symmetric group acting by conjugation. We will summarize the relevant results from [2] in §1 below. We are then in a position to determine the fixed point sets, and the vertices of the components of, the permutation module afforded by the action of $\mathrm{Sym}(2n)$ on its conjugacy class Ξ_{2n} of fixed point free involutions. This permutation action has been studied before; as an example the ordinary character it affords is known, and we refer to [5] for a demonstration of this and other results. With any p-permutation module one can associate an ordinary character, and we will determine these characters for the components of the permutation module on fixed point free involutions. For general results we refer to §26 and §27 in [7].

1. Summary of results from [3]

1.1 Fixed point sets Let G be an arbitrary finite group, Ω some finite G-set and $M = k\Omega$ the resulting permutation module. In [1] the components of M which have a vertex Q are parameterized as follows.

1.2 Theorem *Let Ω be a permutation G-space. There is a multiplicity-preserving 1-1 correspondence between*
(i) components of M with vertex Q, and
(ii) components of $M(Q)$ which are projective as modules for $N_G(Q)/Q$.

Here $M(Q) = k[\mathrm{Fix}(Q)]$, which is a permutation module for the group $N_G(Q)/Q$ in the natural way, and $\mathrm{Fix}_\Omega(Q)$ is the set of fixed points Q in Ω. We will refer to this as 'Broué Correspondence'.

The main object of study in [3] are the subsets of Ω which can occur as fixed point sets. The basic definition is as follows.

(1.3) Definition. *Suppose X is a subset of Ω. Then X is a fixed point set if $X = \mathrm{Fix}_\Omega(Q)$ where Q is a vertex for some component W of $k\Omega$.*

For any subset X of Ω, we write $S_X := \mathrm{Stab}_G(X)$ for the pointwise stabilizer of X, and we write Q_X for some Sylow p-subgroup of S_X.

We say that the subset X of Ω is **closed** if $\mathrm{Fix}_\Omega(Q_X) = X$.

It is easy to see that any fixed point set is closed, furthermore the group Q in the above definition of a fixed point set can be taken as $Q = Q_X$. [See [3] (2.3)] Now let N_X be the set stabilizer of X, and let $\overline{N}_X := N_X/S_X$; then X is a permutation $\overline{N}_X$-set. Then we have the following modification of the Broué Correspondence (but not yet taking multiplicities into account). [See [3] (2.12)]

1.4 Proposition. Assume Ω is a G-set, and $X \subseteq \Omega$ is a closed subset. Then X is a fixed point set if and only if kX as a module for $\overline{N}_X$ has a projective component.

1.5 Fixed point sets by conjugation We consider now a symmetric group acting by conjugation on itself. In this case, G-sets derive additional structure from multiplying permutations. The general setup studied in [3] is as follows.

We consider permutations with finite supports on arbitrary subsets of the natural numbers, that is, let Γ be the group of finitary permutations on $\mathbb{N}$. Let

$$\mathcal{F} := \{X \subset \Gamma, |X| < \infty\} \setminus \{id\}.$$

Most important is the subset $\mathcal{S}^q \subset \mathcal{F}$, where X is an element of $\mathcal{S}^q$ if and only if it consists of permutations which are products of q-cycles. In our application later we will take $q = 2$.

For $X \in \mathcal{F}$, let G_X be the group of all permutations on the support $\mathrm{supp}(X)$. Then we have a natural G_X-set which has X as a subset, namely

$$\Xi_X := \cup_{g \in G_X} X^g.$$

We want to understand when X is a fixed point set, as a subset of G_X-permutation set Ξ_X.

1.6 Multiplicative structure of $\mathcal{F}$ We define an equivalence relation on $\mathcal{F}$. Take X_1 and X_2 in $\mathcal{F}$. We say

$$X_1 \sim X_2 \Leftrightarrow \text{ there is} g \in \Gamma \text{ such that } X_1^g = X_2.$$

We work with equivalence classes but keep the same notation. (See [2] for details.)

Constructions (1) [Products] Let $X, Y \in \mathcal{F}$. Take $g \in \Gamma$ such that X and Y^g have disjoint supports. Define

$$X * Y := \text{ the class containing } X * Y^g.$$

(2) [Powers] Let $X \in \mathcal{F}$. Define $*^2X = X * X$ and inductively

$$*^sX = (*^{s-1}X) * X$$

which is the class of $X \times X^{g_2} \times \cdots \times X^{g_s}$ for $g_i \in \Gamma$ where the fators have disjoint supports. We write X^s for simplicity.
(3) [Diagonals] Let $X \in \mathcal{F}$. Then define $\Delta^s(X)$ to be the class of a subset of $*^sX$. If $*^sX$ is represented by $X \times X^{g_2} \times \cdots \times X^{g_s}$ as in (2), then $\Delta^s(X)$ is the class of the subset

$$\{x * x^{g_2} * \cdots * x^{g_s} | x \in X\}.$$

We say that an element $X \in \mathcal{F}$ is **irreducible** if there are no $Y, Z \in \mathcal{F}$ such that $X = Y * Z$. Then every $X \in \mathcal{F}$ has a factorisation into a product of irreducible sets, and this is essentially unique. Given two elements X and Y in $\mathcal{F}$, we may say that X and Y are **coprime** if they do not share an irreducible factor (upto equivalence).

A set $X \in \mathcal{F}$ is called **exact** if it is fixed point free on the support $\mathrm{supp}(X)$. On the other hand we call an irreducible element X **projective** if $Q_X = 1$, and an element Y projective-free if none of its irreducible factors is projective. It is proved in [2] that if X is an irreducible fixed point set then it is either exact or projective. The second case occurs precisely when X corresponds to a projective component of the permutation module.

Furthermore, a fixed point set is said to be **transitive** if the group $\langle X \rangle$ is transitive on the support of X.

Finally, for $X \in \mathcal{F}$ let the number $\kappa(X)$ be the lowest positive integer u such that the permutation module $kX^{\wr u}$ for $H \wr \mathrm{Sym}(u)$ does not have a projective component, allowing the possibility $\kappa(V) = \infty$ if no such u exists.

With these, the general description of a fixed point set in $\mathcal{S}^q$ given in [2] [Theorem 7.18] is as follows. Let $Y \in \mathcal{S}^q$, write $Y = Y_1^{a_1} * Y_2^{a_2} * \cdots * Y_t^{a_t}$ where the Y_i are pairwise coprime and irreducible.

1.7 Theorem [[3], 7.18] *Assume the setup of before, with elements of $\mathcal{S}^q$. Assume $X \in \mathcal{S}^q$.*
(1) X is an irreducible exact fixed point set $\Leftrightarrow X = \Delta^{p^i} Y$ where Y is a transitive irreducible exact fixed point set and $i \geq 0$.
(2) X is a projective-free fixed point set $\Leftrightarrow X = X_1^{a_1} * \cdots * X_t^{a_t}$ where the X_i are pairwise coprime irreducible exact fixed point set, and $1 \leq a_i < \kappa(X_i)$.
(3) X is a fixed point set $\Leftrightarrow X = W * V$ where W is a projective-free fixed point set and V is an irreducible projective fixed point set. The motivation for considering transitivity is the following (see [2], 7.14). For $Y \in \mathcal{S}^q$ we define the **degree** of Y to be $\mathrm{d}(Y) = |\mathrm{supp}(Y)|$.

1.8 Theorem [[3], 7.14] *Assume $Y \in \mathcal{S}^q$ is an exact fixed point set which is transitive. Then $\mathrm{d}(Y) = p$ or pq.*

This will be sufficient to classify all fixed point sets which occur for components of our fixed point free permutation module in the case $p = q = 2$.

2. The components of $k\Xi_{2n}$

2.1 Let k be a field of characteristic 2 and Ξ_{2n} be the conjugacy class of $\mathrm{Sym}(2n)$ containing the fixed point free involutions. We intend to enumerate the components of the permutation $k(\mathrm{Sym}(2n))$-module $k\Xi_{2n}$, along with the associated vertices and Brauer characters, by enumerating its fixed point sets. As we have mentioned this permutation action has been studied before. The ordinary character afforded by this action has been calculated in general and the paper [5] contains a proof of this result. We quote the following theorem from the same paper which provides an even more general result, which will be useful later when we determine the characters of the components of $k\Xi_{2n}$.

To describe it, we let n and m be non-negative integers. Let sgn_{2m} be the module over $\mathbb{C}$ afforded by the sign representation of $\mathrm{Sym}(2m)$. Then the outer tensor product $\mathbb{C}\Xi_{2n}\#\mathrm{sgn}_{2m}$ is a $\mathrm{Sym}(2n) \times \mathrm{Sym}(2m)$-module, and the group $\mathrm{Sym}(2n) \times \mathrm{Sym}(2m)$ is a Young subgroup of $\mathrm{Sym}(2 \cdot (m+n))$. Let $\Xi_{2n,2m} := \mathrm{Ind}_{\mathrm{Sym}(2n)\times\mathrm{Sym}(2m)}^{\mathrm{Sym}(2(n+m))}(\mathbb{C}\Xi_{2n}\#\mathrm{sgn}_{2m})$ denote the induced $\mathbb{C}\mathrm{Sym}(2(n+m))$-module. If $n = 0$ then put $\Xi_{2n,2m} := \mathrm{sgn}_{2m}$.

Note that we have $\Xi_{2n,0} = \Xi_{2n}$, which is the permutation $\mathrm{Sym}(2n)$-space under consideration.

(2.2) Theorem. Inglis-Richardson-Saxl [5] *Let Λ_{2n}^m be the set of partitions of $2n$ using precisely m odd parts. The character $\chi_{2n,2m}$ of the $\mathbb{C}\mathrm{Sym}(2(n+m))$-module $\Xi_{(2n,2m)}$ satisfies $\chi_{(2n,2m)} = \sum_{\lambda \in \Lambda_{2(n+m)}^{2m}} \chi^\lambda$.*

Proof. A short proof is given in [5]. □

(2.3) Lemma. *For every positive integer n the complex character χ_n afforded by the permutation $\mathrm{Sym}(2n)$-space Ξ_{2n} is given by*

$$\chi_n := \chi_{\Xi_{2n}} = \sum_{\lambda \in \Lambda_{2n}^0} \chi^\lambda.$$

Proof. This is the previous theorem applied to the case $m = 0$. $\square$

(2.4) Theorem. *The permutation $k\mathrm{Sym}(2m)$-module $k\Xi_{2m}$ admits no projective components.*

Proof. This follows from the main theorem of [6] (J. Murray). $\square$

The following lemma presents an elementary result which will help later to describe the fixed point sets of $k(\Xi_{2n})$.

(2.5) Lemma. The natural $k\mathrm{Sym}(3)$-module. *Let the field k be of characteristic $p = 2$ with $G = \mathrm{Sym}(3)$. Let $P =< (1,2) >$ be the Sylow 2-subgroup of G generated by the transposition $(1,2)$. There is a natural action of G on the set $X := \{1,2,3\}$ which affords a permutation kG-module kX, called the natural $k\mathrm{Sym}(3)$-module. This module admits a projective component.*

Proof. Since $|X| = 3$ is coprime to 2 we have $kX = 1_{kG} \oplus E$ is the direct sum of the trivial kG-module and the submodule E spanned over k by the set $Y := \{1+3, 2+3\}$. The submodule E is termed the standard $k\mathrm{Sym}(3)$-module. The dimension of E is 2. Note that P acts regularly on Y and this ensures that E is a projective kP-module, from which it follows that E is a projective kG-module. $\square$

2.6 In light of the fact that (2.4) shows that there are no projective components of Ξ_{2n}, the problem of finding the fixed point sets reduces by (1.7) to finding the transitive fixed point sets.

By (1.8), any transitive fixed point set has support of size 2 or 4. We can now find these sets.

The singleton set Let $U := \{(12)\}$. This is a fixed point set: the group G_U is the symmetric group of degree 2 and $U = \Xi_2$. We have $S_U = G_U = Q_U$ and kU is the trivial module which has vertex Q_U. This is the only way to have a fixed point set with support of size 2.

Transitive irreducible fixed point sets supported on 4 elements We must analyze the permutation $\mathrm{Sym}(4)$ module afforded by the permutation space $\Xi_4 = \{(12)(34), (13)(24), (14)(23)\}$. The permutation module is isomorphic to $\mathrm{Ind}_D^G(k)$ where D is the stabilizer of $(12)(34)$ (say) in $G = \mathrm{Sym}(4)$. So D is dihedral of order 8. This group D contains the Klein 4-group $H = \mathrm{id} \cup \Xi_4$ which is normal in G, and H acts trivially on Ξ_4. Viewed as a module for the factor group $G/H \cong \mathrm{Sym}(3)$, the module $k\Xi_4$ is the natural 3-dimensional permutation module. As explained in (2.5) we have $k\Xi_4 = k \oplus E$ where E is projective as a module for $\mathrm{Sym}(3)$. The trivial summnad has vertex D, and D has fixed points just the singleton set $\{(12)(34)\}$. This is equivalent to U^2 which is not transitive or irreducible.

The module E has vertex H as a module for G, and $\mathrm{Fix}(H) = \Xi_4$. So Ξ_4 is a fixed point set, and it is irreducible and transitive.

(2.7) Theorem. Fixed Point Sets of the Family $k(\Xi_{2n})$. *Write $U := \Xi_2$, $V := \Xi_4$ and $V_i := \Delta^{p^i} v$. Then a complete list without repetitions of irreducible fixed point sets is $U, V_0, V_1, V_2, \ldots$. So if $X \subset \Xi_{2n}$ for some value of n then X is an irreducible fixed point set if and only if $X = U$ or $X = V_i$ for some positive integer i.*

Proof. By (1.8) any transitive fixed point set has degree 2 or 4. By (2.6) then the

only transitive fixed point sets are Ξ_2 or Ξ_4. Let us put $U := \Xi_2$ and $V := \Xi_4$. Now let X be any irreducible fixed point set. By (1.7) the set X satisfies $X = \Delta^{p^i} U$ or $X = \Delta^{p^i} V$, for some positive integer i. But $|U| = 1$ so $\Delta^{p^i} U = *^{p^i} U$ is not irreducible. Hence either $X = U$ or $X = \Delta^{p^i} V$ for some i. Let $V_i := \Delta^{p^i} V$. Then V_i *is* an irreducible fixed point set by (1.7). Let us write $V_0 := V$. Then a complete list of the irreducible fixed point sets is $U, V_0, V_1, V_2, \dots$. □

(2.8) Theorem. *Let $U := \Xi_2$ and let $V_i := \Delta^{2^i} \Xi_4$. Let I be a finite set of non-negative integers and s a non-negative integer. Then $U^s * (*_{i \in I} V_i)$ is a fixed point set, assuming that either $s \neq 0$ or $I \neq \emptyset$. Furthermore any fixed point set has this form.*

Proof. First we observe that the singleton set $U^s := (\Xi_2)^s = \{(1\ 2)(3\ 4) \cdots (2s - 1\ 2s)\}$ is always a fixed point set and corresponds to the trivial component of $k\Xi_{2s}$, which can be seen directly.

The sets V_i are pairwise distinct irreducible fixed points, hence pairwise coprime, and each is coprime to U^s.

It follows from (1.7) that $U^s * (*_{i \in I} V_i)$ *is* a fixed point set. Furthermore every fixed point set is of the form $U^s * (*_{i \in I} V_i^{a_i})$ for non-negative integers s and $a_i \in I$ for some set I satisfying $a_i < \kappa(V_i)$. It is not hard to show that $\kappa(V_i) = 1$ for each i. □

(2.9) Lemma. *Let $X, Y, Z \in \mathcal{F}$ and assume X irreducible. Then the pair $(\overline{N}_{\Delta^t X}, \Delta^t X)$ is permutation isomorphic to the pair $(\overline{N}_X, X)$. Furthermore if Y and Z are coprime then $N_{Y*Z} = N_Y \times N_Z$.*

Proof. See [3] (4.9) and (5.4). □

2.10 Brauer Quotients and the Broué Correspondence. We have determined the irreducible fixed point sets of $k(\Xi_{2m})$. Every such set W determines a vertex Q_W and gives rise to a Brauer quotient. We recall that this Brauer quotient is the pair consisting of the factor group $N_{\mathrm{Sym}(2m)}(Q_W)/Q_W$ and the permutation module kW on which the factor group acts. The Broué correspondence (1.2) tells us that the set of projective components of kW is in 1-1 correspondence (preserving multiplicities) with the components of $k(\Xi_{2m})$ with vertex Q_W. We will later show that these sets are singletons. The group $\overline{N}_W := N_W/S_W$ also acts on the set W, and in fact the projective components of the module afforded are in 1-1 correspondence with the projective components of the Brauer quotient. This follows because the group $\overline{N}_W$ is isomorphic to the quotient $(N_{\mathrm{Sym}(2m)}(Q_W)/Q_W)/T$ where $T = N_{S_W}(Q_W)/Q_W$ is a normal $2'$ subgroup which acts trivially on W. The components of kW as a $k\overline{N}_W$ module therefore inflate to give the components of kW as a $N_{\mathrm{Sym}(2m)}(Q_W)/Q_W$ module, and since T is $2'$ this preserves projectivity. For this reason we will refer to the pair $(\overline{N}_W, kW)$ as the equivalent of the Brauer quotient of $k(\Xi_{2m})$ with respect to Q_W. See [3] (2.11) for more details.

Consequently it will be sufficient for us to describe the permutation module given by the pair $(\overline{N}_W, W)$, in the sense that the Broué correspondence extends to a 1-1 multiplicity preserving correspondence between the set of components of $k(\Xi_{2m})$ with vertex Q and the set of projective components of the $k\overline{N}_W$ module kW. We do this now for the irreducible fixed points sets, giving the general result in the theorem that follows.

Firstly consider $U = \{(1\ 2)\}$ and $\overline{N}_U = \mathrm{Sym}(1\ 2)/\mathrm{Sym}(1\ 2) \cong \mathrm{id}$. Here $\overline{N}_U$ is the trivial group and kU is the trivial $k\overline{N}_U$-module. In other words the pair $(\overline{N}_U, U)$ is permutation isomorphic to the pair $(1, 1)$.

Now let i be a non-negative integer and consider the permutation $\overline{N}_{V_i}$-space V_i. By (2.9) pair $(\overline{N}_{V_i}, V_i)$ is permutation isomorphic to the pair $(\overline{N}_V, V)$. Now V is the set of double transpositions in $\mathrm{Sym}(4)$ so $N_V := N_{\mathrm{Sym}(4)}(V) = \mathrm{Sym}(4)$. Furthermore $S_V := \mathrm{Stab}_{\mathrm{Sym}(4)}(V) = H$, where H denotes the Klein 4-subgroup. Thus $\overline{N}_V \cong \mathrm{Sym}(3)$ and we are in the situation of (2.5). In particular the pair $(\overline{N}_V, V)$ is permutation isomorphic to the pair $(\mathrm{Sym}(3), X)$, where X denotes the natural permutation $\mathrm{Sym}(3)$-space. We saw in (2.5) that $kN = k \oplus kE$, the summand kU being indecomposable projective.

(2.11) Theorem. *Let I be a finite set of non-negative integers with $t := |I|$. Let s be a non-negative integer and put $W := U^s * (*_{i\in I} V_i)$. Then $\overline{N}_W$ is isomorphic to the direct product of t copies of* $\mathrm{Sym}(3)$. *After this identification the permutation $k\overline{N}_W$-module kW satisfies $kW \cong \bigotimes^t kX$, where X is the natural permutation* $\mathrm{Sym}(3)$*-space.*

Proof. The set U^s is a singleton set and so we have $N_{U^s} = S_{U^s}$, and $\overline{N}_{U^s} = 1$. Thus the pair $(\overline{N}_{U^s}, U^s)$ is permutation isomorphic to the pair $(1, 1)$. By (2.9) for any $i \in I$ the pair $(\overline{N}_{V_i}, V_i)$ is permutation isomorphic to the pair $(\mathrm{Sym}(3), X)$. The pairwise coprimeness of the irreducible factors of W shows that $N_W = N_{U^s} \times \prod_{i\in I} N_{V_i}$, from which the result follows. □

The aim now is to find the corresponding description of the vertices.

(2.12) Convention. Whenever W is a fixed point set, assume that W has been chosen so that $W \subset \Xi_{2n}$, where $n := d(W)/2$, so that the support of W is the set $\{1, 2, \dots, 2n\}$. This means the subgroup Q_W is a vertex of some component of $k(\Xi_{2n})$, by (1.3).

2.13 Notation. Fix a positive integer n and consider the permutation $k\mathrm{Sym}(2n)$-module $k(\Xi_{2n})$. Let $\mu := (s, t)$ be a pair of non-negative integers satisfying $s + 2t = n$. If $t = 0$ put $I := \emptyset$. Otherwise let $t = \sum_{l=1}^{r} 2^{i_l}$ be the 2-adic expansion of t and let $I := \{i_l\}_{l=1}^{r}$ be the set of exponents used. Define $W_\mu := U^s * (*_{i\in I} V_i)$ and $Q_\mu := Q_{W_\mu}$, $N_\mu := N_{W_\mu}$ and $\overline{N}_\mu := \overline{N}_{W_\mu}$. Then with the above convention in force the 2-subgroup Q_μ is a vertex of some component of $k(\Xi_{2n})$.

The size of the support of W_μ is given by $2n = |\mathrm{supp}(W_\mu)| = 4t + 2s$ where $4t = \sum |\mathrm{supp}(V_i)|$. Since each V_i has support of size a power of 2, the possible W_μ are (by uniqueness of 2-adic expansion) in 1-1 correspondence with solutions in non-negative integers of the equation $4t + 2s = 2n$.

(2.14) Theorem. *Fix a positive integer n and consider the permutation* $k\mathrm{Sym}(2n)$*-module $k(\Xi_{2n})$. Define the set $\mathcal{Q} := \{Q_\mu\}$ over all the $\lfloor n\ 2\rfloor + 1$ pairs $\mu = (s, t)$ that are non-negative integer solutions to $4t + 2s = 2n$. Then $\mathcal{Q}$ is a set of $\lfloor n/2\rfloor + 1$ pairwise non-isomorphic 2-subgroups of* $\mathrm{Sym}(2n)$. *Furthermore every element $Q \in \mathcal{Q}$ is a vertex of exactly one component of $k(\Xi_{2n}^2)$. In particular the number of such components is $\lfloor n/2\rfloor + 1$.*

Proof. By (1.7) the general form of a fixed point set $W = U^e * (*_{j\in J} V_j)$. Here e is a non-negative integer and J is a set of non-negative integers,

and either $e > 0$ or $J \neq \emptyset$. The degree of W is given by

$$d(W) = |supp\ (U)| \cdot e + \sum_{j \in J} |\text{supp}\ (V_i)| = 2e + \sum_{j \in J} 2^j \cdot 4.$$

With our convention in force, the fixed point set W satisfies $W \subset \Xi_{2n}$ if and only if $d(W) = 2n$, and this is the case exactly when $n = e + 2 \cdot (\sum_{j \in J} 2^j)$. It follows that the set $\mathcal{Q}$ of vertices of components of $k(\Xi_{2n})$ is complete.

Suppose that the pairs $\mu = (4t, 2s)$ and $\lambda = (4t_1, 2s_1)$ of non-negative integers satisfy $2s + 4t = 2s_1 + 4t_1 = 2n$. Suppose also that μ and λ are unequal. Then s is the number of orbits on $\text{supp}(W_\mu)$ which are of size 2 under the action of $\langle W_\mu \rangle$. The corresponding remark holds for s_1 and so W_μ and W_λ cannot be conjugate in $\text{Sym}(2n)$. But the sets W_μ and W_λ are closed so they satisfy $W_\mu = \text{Fix}\ Q_\mu$ and $W_\lambda = \text{Fix}\ Q_\lambda$. Thus Q_μ and Q_λ are non-conjugate subgroups of $\text{Sym}(2n)$.

To see that the subgroup Q_μ is a vertex of no more than one component of $(k\Xi_{2n})$ we consider the permutation $k\overline{N}_{W_\mu}$-module kW_μ. By (2.11) the pair $(\overline{N}_{W_\mu}, W_\mu)$ is permutation isomorphic to the pair $(\text{Sym}(3)^t, \bigotimes^t kX)$. By (2.5) the permutation $k\text{Sym}(3)$-module kX admits the unique component projective kE, where U is the standard permutation $\text{Sym}(3)$-space. A standard result now implies that $\bigotimes^t kX$ admits the unique projective component $\bigotimes^t kE$. So by the Broué correspondence as given in (2.10) Q_μ is a vertex of exactly one component. $\square$

2.15 Vertices For each composition μ of $2n$ of the form $\mu = (4t, 2s)$ we have the fixed point set $W_\mu = (*_{i \in I} V_i) * U^s$ where $t = \sum_{i \in I} 2^i$ is the 2-adic expansion. Then by the results of [2] a component which corresponds to such a fixed point set has vertex a Sylow 2-subgroup of the pointwise stabilizer S_{W_μ} of W_μ. Since the factors of W_μ are coprime, this stabilizer is just the product of the stabilizers of the factors (by (2.9)) so that

$$S_{W_\mu} = S_{U^s} \times (\prod_{i \in I} S_{V_i})$$

In the following denote by $P(b)$ a Sylow 2-subgroup of $\text{Sym}(b)$. It follows that
(a) Q_{U^s} is a Sylow 2-subgroup $P(2s)$ of $\text{Sym}(2s)$ (since S_{U^s} is the centralizer of a fixed point free involution in $\text{Sym}(2s)$.)

We have (see [2], Lemma (4.5)) that $S_{V_i} \cong S_V \wr \text{Sym}(2s)$. Now $S_V = H$, the Klein 4-group on four points, $\Xi_4 \subset H$, and hence
(b) $Q_{V_i} = H \wr P(2^i)$, on $4 \cdot 2^i$ points. Combining these, a vertex of a component with fixed point set W_μ where $\mu = (4t, 2s) \models 2n$ is of the form

$$P(2s) \times (\prod_{i \in I} (H \wr P(2^i))$$

where t has 2-adic expansion $y = \sum_{i \in I} 2^i$.

3. The ordinary characters of the components

3.1 So far we have determined the isomorphism type of the components of $k\Xi 2n$ by identifying their vertices and Brauer quotients. We now turn our attention to computing the character of the complex lift of these components. Let $\mathcal{O}$ be a complete discrete valuation ring with residue field k and field of fractions K; then the components of $k\Xi_{2n}$ lift to components of $\mathcal{O}\Xi_{2n}$, and the required character of

a component M of $\mathcal{O}\Xi_{2n}$ is the ordinary character of the KG-module $K \otimes M$. In the following we fix n and write $G = \mathrm{Sym}(2n)$.

The lifts of the components of $k\Xi_{2n}$ have the same vertices as the mod-2 reductions, so we can keep the labelling:

(3.2) Lemma. *The permutation $\mathcal{O}\mathrm{Sym}(2n)$-module $\mathcal{O}\Xi_{2n}$ has exactly $\lfloor n/2 \rfloor + 1$ components, labelled as M_μ where μ runs through the compositions of $2n$ of the form $\mu = (4t, 2s)$ with $t, s \geq 0$. The vertices of these components are pairwise non-conjugate in G. Let Q_μ be the vertex of M_μ. Then Q_μ is contained in $\mathrm{Alt}(2n)$ if and only if if $\mu = (4t, 0)$. If so then n is even.*

Proof. See (2.15). $\square$

Notation By our labelling, the component M_μ corresponds to the fixed point set $W_\mu = (*_{i \in I} V_i) * U^s$; and the vertex Q_μ of M_μ is then a Sylow 2-subgroup of the stabilizer of W_μ. By coprimeness, there is a factorization

$$Q_\mu = Q_{(4t,0)} \times P(2s)$$

where $P(2s)$ is a Sylow 2-subgroup of $\mathrm{Sym}(2s)$ and $Q_{(4t,0)}$ is a direct product of subgroups of the form $H \wr P(2^i)$, as in (2.15).

(3.3) Lemma. *Denote by* alt *the alternating module, then*

$$M_{(4t,0)} \otimes alt \cong M_{(4t,0)}.$$

Proof. Write $\nu = (4t, 0)$. Let X be a Q-invariant basis of $M(\nu)$. The vertex Q_ν of M_ν is contained in $\mathrm{Alt}(4t)$ so

$$M_\nu \otimes \mathrm{alt} \,|\mathrm{Ind}_Q^{\mathrm{Sym}(4t)}(k) \otimes \mathrm{alt} \cong \mathrm{Ind}_Q^{\mathrm{Sym}(4t)}(k \otimes \mathrm{alt}) = \mathrm{Ind}_Q^{\mathrm{Sym}(4t)}(k).$$

Thus $M_\nu \otimes \mathrm{alt}$ is a p-permutation $k\mathrm{Sym}(4t)$-module with vertex Q_ν. It will suffice to show that this module has the same Brauer quotient (in the sense of (2.10)) as M_ν. Now Q only contains even permutations so $\{x \otimes 1 : x \in X\}$ is a Q-invariant basis of $M_\nu \otimes \mathrm{alt}$. Thus as $\mathcal{O}N_{2n}$-modules we have

$$(M_\nu \otimes \mathrm{alt})(Q) \cong M_\nu(Q) \otimes \mathrm{alt}.$$

The images in $\overline{N}_\nu \cong \mathrm{Sym}(3) \times \mathrm{Sym}(3) \cdots \times \mathrm{Sym}(3)$ of odd permutations of N_ν are again odd permutations. Therefore as $\mathcal{O}\overline{N}_{2n}$-modules we have

$$(M_\nu \otimes \mathrm{alt})(Q) \cong M_\nu(Q) \otimes \mathrm{alt}.$$

Now $M_\nu(Q) = E \# E \# \cdots \# E$ so

$$(M_\nu \otimes \mathrm{alt})(Q) \cong (E \# E \# \cdots \# E) \otimes \mathrm{alt} \cong (E \otimes \mathrm{alt}) \# (E \otimes \mathrm{alt}) \# \cdots \# (E \otimes \mathrm{alt}).$$

But $E \otimes \mathrm{alt} \cong E$, as may be seen by considering characters, and so $(M_\nu \otimes \mathrm{alt})(Q) \cong M_\nu(Q)$. $\square$

Write $\overline{M}_\mu$ for the kG-module $k \otimes_{\mathcal{O}} M_\mu$.

(3.4) Lemma. *Let $H = \mathrm{Sym}(4t) \times \mathrm{Sym}(2s) \leq G$. Then $N_G(Q_{(4t,2s)}) \leq H$ and the kG-module $\overline{M}_{(4t,2s)}$ is the Green correspondent of the kH-module $\overline{M}_{(4t,0)} \# k$.*

Proof. A vertex Q of $M_{(4t,2s)}$ can be taken to be $Q_{(4t,2s)}$, recalling the notation of (2.13). By (2.8) the set of fixed points of Q satsifies $X = Y * Z$ for coprime Y and

Z such that $Y * Z \subset H$. So $N_X \leq H$ by (2.9), and we must have $N_G(Q) \leq H$ too, since $N_G(Q) \leq N_X$.

We may therefore let the $\mathcal{O}H$-module L be the Green correspondent of $M_{(4t,2s)}$. Each of $M_{(4t,2s)}, L$ and $M_{(4t,0)}\#k$ has vertex Q, the latter by (2.15). The Brauer quotient of $M_{(4t,2s)}$, and hence also of L, is, by (2.10), the $k(N/Q)$-module $M_{(4t,0)}(Q)\#k$, where $M_{(4t,0)}(Q)$ denotes the Brauer quotient of $M_{(4t,0)}$. This implies that $L \cong M_{(4t,0)}$ and the result follows. $\square$

(3.5) Lemma. *Let $2n = 4t + 2s$. The $k\mathrm{Sym}(2n)$-module $M_{(4t,2s)}$ is a component of $\mathrm{Ind}_H^{\mathrm{Sym}(2n)}(M_{(4t,0)}\#k)$.*

Proof. This follows immediately from the preceding lemma. $\square$

(3.6) Corollary. *Let $2n = 4t+2s$. The module $M_{(4t,2s)} \otimes \mathrm{alt}$ is a component of $\mathrm{Ind}_H^{\mathrm{Sym}(2n)}(k\Xi_{2n}\#\mathrm{alt})$.*

Proof. By the previous lemma $M_{(4t,2s)} \otimes \mathrm{alt}$ is a component of

$$(\mathrm{Ind}_H^{\mathrm{Sym}(2n)}(M_{(4t,0)}\#k)) \otimes \mathrm{alt} \cong \mathrm{Ind}_H^{\mathrm{Sym}(2n)}((M_{(4t,0)} \otimes \mathrm{alt})\#\mathrm{alt});$$

and by (3.3) this module is isomorphic to

$$\mathrm{Ind}_H^{\mathrm{Sym}(2n)}(M_{(4t,0)}\#\mathrm{alt}),$$

and the result is implied. $\square$

(3.7) Corollary. *Every irreducible character contained in the decompostion of the ordinary character of the module $M_{(4t,2s)}$ is indexed by a partition whose conjugate contains exactly $2s$ odd parts.*

Proof. The previous corollary along with Theorem (2.2) tells us that the conjugate of the module $M_{(4t,2s)}$ has an ordinary character for which each component is indexed by a partition with exactly $2s$ odd parts. The claim follows directly. $\square$

(3.8) Theorem. *Let $2n = 4t + 2s$. The module $M_{(4t,2s)}$ has character $\phi_{(4t,2s)}$ given by*

$$\phi_{(4t,2s)} = \sum_{\lambda \in \Lambda_{2n}^0 \cap \Lambda_{2n}'^{2s}} \chi^\lambda,$$

where $\Lambda_v'^u$ is the set of partitions of v whose conjugates have exactly u odd parts. In other words, the module $M_{(4t,2s)}$ has character equal to the sum of all irreducible characters of $\mathrm{Sym}(2n)$ that are indexed by a partition that has no odd parts but whose conjugate contains exactly $2s$ odd parts.

Proof. By (2.14) each module $M_{(4r,2n-4r)}$ occurs with multiplicity one as an component of $k(\Xi_{2m})$. Therefore by (2.3) we have the following equality of characters:

$$\chi_{\Xi_{2n}} = \sum_{\lambda \in \Lambda_{2n}^0} \chi^\lambda = \sum_{r=0}^{\lfloor n/2 \rfloor} \phi_{(4r,2n-4r)}.$$

However, the preceding corollary tells us that $M_{(4r,2n-4r)}$ is a component of a module whose character is, by applying (2.2) to its conjugate, given by:

$$\chi_{(4r,2n-4r)} := \sum_{\lambda \in \Lambda_{2n}'^{(2n-4r)}} \chi^\lambda.$$

In other words there is a subset $\Omega_{2n}^{\prime(2n-4r)}$ of $\Lambda_{2n}^{\prime(2n-4r)}$ such that

$$\phi_{(4r,2n-4r)} = \sum_{\lambda \in \Omega_{2n}^{\prime(2n-4r)}} \chi^\lambda$$

so that

$$\sum_{\lambda \in \Lambda_{2n}^0} \chi^\lambda = \sum_{r=0}^{\lfloor n/2 \rfloor} \phi_{(4r,2n-4r)} = \sum_{r=0}^{\lfloor n/2 \rfloor} \sum_{\lambda \in \Omega_{2n}^{\prime(2n-4r)}} \chi^\lambda.$$

The sets $\Lambda_{2n}^{\prime(2n-4r)} \cap \Lambda_{2n}^0$ partition Λ_{2n}^0 , and since every $\Omega_{2n}^{\prime(2n-4r)}$ is a subset of $\Lambda_{2n}^{\prime(2n-4r)} \cap \Lambda_{2n}^0$ we must have $\Omega_{2n}^{\prime(2n-4r)} = \Lambda_{2n}^{\prime(2n-4r)} \cap \Lambda_{2n}^0$, completing the proof. $\Box$

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